

AN INVARIANCE PRINCIPLE FOR STATIONARY RANDOM FIELDS UNDER HANNAN'S CONDITION

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ABSTRACT. We establish an invariance principle for a general class of stationary random fields indexed by \mathbb{Z}^d , under Hannan's condition generalized to \mathbb{Z}^d . To do so we first establish a uniform integrability result for stationary orthomartingales, and second we establish a coboundary decomposition for certain stationary random fields. At last, we obtain an invariance principle by developing an orthomartingale approximation. Our invariance principle improves known results in the literature, and particularly we require only finite second moment.

1. INTRODUCTION

Let $\{X_i\}_{i \in \mathbb{Z}^d}$ be a stationary random field with zero mean and finite variance, and let S_n be the partial sum with $n \in \mathbb{N}^d$

$$S_n = \sum_{1 \leq i \leq n} X_i,$$

and we are interested in the invariance principle of normalized partial sums in form of

$$(1.1) \quad \left\{ \frac{S_{[n \cdot t]}}{(n_1 \cdots n_d)^{1/2}} \right\}_{t \in [0,1]^d} \Rightarrow \{\sigma \mathbb{B}_t\}_{t \in [0,1]^d},$$

where $n \cdot t = (n_1 t_1, \dots, n_d t_d)$. We provide a sufficient condition for the above weak convergence to hold in $D[0,1]^d$, and the limiting random field is a Brownian sheet.

The invariance principle for standard Brownian sheet has a long history, and people have investigated this problem from different aspects. See for example Berkes and Morrow [2], Bolthausen [3], Goldie and Morrow [12], Bradley [4] for results under mixing conditions, Basu and Dorea [1], Nahapetian [19], Poghosyan and Røelly [21] for results on multiparameter martingales, and Dedecker [6, 7], El Machkouri et al.

2010 *Mathematics Subject Classification.* Primary, 60F17, 60G60; secondary, 60F05, 60G48.

Key words and phrases. Invariance principle, Brownian sheet, random field, orthomartingale, Hannan's condition, weak dependence.

[11], Wang and Woodroffe [23] for results on random fields satisfying projective-type assumptions. In particular, projective-type assumptions have been significantly developed for invariance principles for stationary sequences ($d = 1$). See for example Wu [24], Dedecker et al. [8], among others, for some recent developments. However, extending these criteria in one dimension to high dimensions is not a trivial problem.

Our goal is to establish a random-field counterpart of the invariance principle for regular stationary sequences satisfying Hannan's condition [15]. Hannan's condition consists of assuming, in dimension one,

$$(1.2) \quad \sum_{i \in \mathbb{Z}} \|P_0(X_i)\|_2 < \infty,$$

where $P_0(X_i) = \mathbb{E}(X_i \mid \mathcal{F}_0) - \mathbb{E}(X_i \mid \mathcal{F}_{-1})$ is the projection operator, with respect to certain filtration $\{\mathcal{F}_k\}_{k \in \mathbb{Z}}$ associated to the stationary sequence $\{X_k\}_{k \in \mathbb{Z}}$. Under Hannan's condition, if in addition the stationary sequence $\{X_n\}_{n \in \mathbb{N}}$ is regular (i.e. $\mathbb{E}(X_0 \mid \mathcal{F}_{-\infty}) = 0$ and X_0 is \mathcal{F}_{∞} -measurable), then the invariance principle follows. Hannan [15] first considered the invariance principle, under the assumption that $\{X_k\}_{k \in \mathbb{Z}}$ is adapted and weakly mixing. The general case for regular sequences was established by Dedecker et al. [8, Corollary 2]. The quenched invariance principle for adapted case has been established by Cuny and Volný [5].

We first generalize the Hannan's condition (1.2) to high dimension. For this purpose we need to extend the notion of the projection operator (Section 2). In particular, we focus on stationary random fields in form of

$$(1.3) \quad X_i = f \circ T_i(\{\epsilon_k\}_{k \in \mathbb{Z}^d}), i \in \mathbb{Z}^d,$$

where $f : \mathbb{R}^{\mathbb{Z}^d} \rightarrow \mathbb{R}$ is a measurable function, T_i is the shift operator on $\mathbb{R}^{\mathbb{Z}^d}$ and $\{\epsilon_k\}_{k \in \mathbb{Z}^d}$ is a collection of independent and identically distributed (i.i.d.) random variables.

Our main result (Theorem 5.1) states if $\mathbb{E}X_0 = 0$, $\mathbb{E}(X_0^2) < \infty$, and the Hannan's condition holds

$$\sum_{i \in \mathbb{Z}^d} \|P_0 X_i\|_2 < \infty,$$

for some projection operator P_0 to be defined (see (2.2) below), then the invariance principle (1.1) holds.

We establish the invariance principle directly, through an approximation by *orthomartingales*. As a consequence, this entails the central

limit theorem in form of

$$\frac{S_n}{|n|^{1/2}} \Rightarrow \mathcal{N}(0, \sigma^2).$$

Our central limit theorem and invariance principle both improve results established in El Machkouri et al. [11] and Wang and Woodroffe [23]. For the central limit theorem, our assumption on the weak dependence, the Hannan's condition, is weaker than theirs. Furthermore, to establish invariance principle we require only finite second moment instead of $2 + \delta$ moment. However, we consider only rectangular index sets as in Wang and Woodroffe [23], while Dedecker [7] and El Machkouri et al. [11] consider more general index sets.

The paper is organized as follows. The basic of orthomartingales is reviewed in Section 2. A uniform integrability result on orthomartingales is established in Section 3, which immediately entails tightness of stationary orthomartingales under finite second moment. Next, an orthomartingale coboundary decomposition is developed in Section 4. At last, the invariance principle under Hannan's condition is established in Section 5. Comparison to related works are provided in Section 6.

2. NOTATIONS AND PRELIMINARIES

We consider partial sums over rectangular sets. For this purpose we write $n = (n_1, \dots, n_d) \in \mathbb{N}^d$ and by $n \rightarrow \infty$ we mean $n_q \rightarrow \infty$ for all $q = 1, \dots, d$. Throughout, for elements in \mathbb{R}^d , operations (including $<, \leq, >, \geq, \pm, \wedge, \vee$) are defined in the coordinate-wise sense. We write $[m, n] = \{i \in \mathbb{Z}^d : m \leq i \leq n\}$ for $m, n \in \mathbb{Z}^d$ and $[n] = [1, n]$. At last, we let $e_q = (0, \dots, 0, 1, 0, \dots, 0)$, $q = 1, \dots, d$ denote canonical unit vectors in \mathbb{R}^d . Write $|n| = \prod_{q=1}^d n_q$. Throughout, let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space.

We first review orthomartingales, essentially following Khoshnevisan [17, Chapter 1.3], and introduce the projection operators. These two concepts are based on the notion of commuting filtrations. Specific examples via commuting transformations are given at the end.

Definition 2.1. A collection of σ -algebras $\{\mathcal{F}_i\}_{i \in \mathbb{Z}^d}$ is a commuting filtration, if $\mathcal{F}_i \subset \mathcal{F}_j$ for all $i, j \in \mathbb{Z}^d$, $i \leq j$ and for all $k, \ell \in \mathbb{Z}^d$ and for all bounded \mathcal{F}_ℓ -measurable random variable Y ,

$$\mathbb{E}(Y \mid \mathcal{F}_k) = \mathbb{E}(Y \mid \mathcal{F}_{k \wedge \ell}), \text{ almost surely.}$$

For the sake of simplicity, we omit 'almost surely' when talking about conditional expectations in the sequel. Given a commuting filtration

$\{\mathcal{F}_i\}_{i \in \mathbb{Z}^d}$, the corresponding filtration $\mathcal{F}^{(q)} = \{\mathcal{F}_\ell^{(q)}\}_{\ell \in \mathbb{Z}}$ defined by

$$\mathcal{F}_\ell^{(q)} = \bigvee_{i \in \mathbb{Z}^d, i_q \leq \ell} \mathcal{F}_i, \quad \ell \in \mathbb{Z}, \quad q = 1, \dots, d,$$

are commuting in the following sense, for all permutation π of $\{1, \dots, d\}$ and bounded random variable Y ,

$$(2.1) \quad \mathbb{E} \left\{ \dots \mathbb{E} \left[\mathbb{E} \left(Y \mid \mathcal{F}_{i_{\pi(1)}}^{(\pi(1))} \right) \mid \mathcal{F}_{i_{\pi(2)}}^{(\pi(2))} \right] \dots \mid \mathcal{F}_{i_{\pi(d)}}^{(\pi(d))} \right\} = \mathbb{E}(Y \mid \mathcal{F}_i),$$

for all $i \in \mathbb{Z}^d$ almost surely [17, p. 36, Corollary 3.4.1].

Given a commuting filtration $\{\mathcal{F}_i\}_{i \in \mathbb{Z}^d}$, we have $\mathcal{F}_i = \bigvee_{j \leq i} \mathcal{F}_j$, and this can be naturally extended to $i \in (\mathbb{Z} \cup \{\infty\})^d$. For example, $\mathcal{F}_\ell^{(1)} = \mathcal{F}_{\ell, \infty, \dots, \infty}$. A commuting filtration $\{\mathcal{F}_i\}_{i \in \mathbb{Z}_+^d}$ with $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$ is defined similarly. To simplify the notation, we write

$$\mathbb{E}_j(\cdot) = \mathbb{E}(\cdot \mid \mathcal{F}_j), j \in \mathbb{Z}^d \text{ and } \mathbb{E}_\ell^{(q)}(\cdot) = \mathbb{E}(\cdot \mid \mathcal{F}_\ell^{(q)}), q = 1, \dots, d, \ell \in \mathbb{Z}.$$

Definition 2.2. A collection of random variables $\{M_n\}_{n \in \mathbb{N}^d}$ is said to be an orthomartingale with respect to a commuting filtration $\{\mathcal{F}_i\}_{i \in \mathbb{Z}_+^d}$, if for all $n \in \mathbb{N}^d$, M_n is \mathcal{F}_n -measurable, $\mathbb{E}|M_n| < \infty$, and

$$\mathbb{E}(M_j \mid \mathcal{F}_i) = M_i \text{ for all } i, j \in \mathbb{Z}_+^d, i \leq j,$$

(we set $M_n \equiv 0$ if $\min\{n_1, \dots, n_d\} = 0$).

Equivalently, given a commuting filtration $\{\mathcal{F}_i\}_{i \in \mathbb{Z}_+^d}$, a collection of random variables $\{M_n\}_{n \in \mathbb{N}^d}$ form an orthomartingale if for each $q = 1, \dots, d$, $\{n_\ell\}_{\ell \neq q} \subset \mathbb{N}$, $n_q \mapsto M_n$ is a one-parameter martingale with respect to the filtration $\mathcal{F}^{(q)}$ [17, p. 37, Theorem 3.5.1]. That is,

$$\mathbb{E}_{n_q-1}^{(q)} M_n = M_{n-e_q} \text{ for all } n \in \mathbb{N}^d, q = 1, \dots, d.$$

Given an orthomartingale $\{M_n\}_{n \in \mathbb{N}^d}$ with respect to a commuting filtration $\{\mathcal{F}_i\}_{i \in \mathbb{Z}_+^d}$, it can be represented as

$$M_n = \sum_{i \in [n]} D_i, n \in \mathbb{N}^d$$

for some $\{D_n\}_{n \in \mathbb{N}^d}$, which are referred to as the orthomartingale differences. When $\{D_n\}_{n \in \mathbb{N}^d}$ is strictly stationary, we say the orthomartingale is stationary. Clearly, for all $n \in \mathbb{N}^d$, D_n is \mathcal{F}_n -measurable and

$$\mathbb{E}_{n_q-1}^{(q)} D_n = 0, q = 1, \dots, d.$$

Finally, we introduce the *projection operators* with respect to a commuting filtrations $\{\mathcal{F}_i\}_{i \in \mathbb{Z}^d}$ defined by

$$(2.2) \quad P_j = \prod_{q=1}^d P_{j_q}^{(q)}, j \in \mathbb{Z}^d$$

with

$$(2.3) \quad P_\ell^{(q)} f = \mathbb{E}_\ell^{(q)} f - \mathbb{E}_{\ell-1}^{(q)} f, \ell \in \mathbb{Z}, q = 1, \dots, d.$$

Lemma 2.3. *Let $\{\mathcal{F}_i\}_{i \in \mathbb{Z}^d}$ be a commuting filtration satisfying (2.1) and $P_\ell^{(q)}$ and P_j be defined as above. Then,*

- (i) $\{P_\ell^{(q)}\}_{\ell \in \mathbb{Z}, q=1, \dots, d}$ are commuting operators, and so are $\{P_j\}_{j \in \mathbb{Z}^d}$.
- (ii) For all $f, g \in L^2(\mathcal{F})$, $\mathbb{E} P_i(f) P_j(g) = 0$ for all $i, j \in \mathbb{Z}^d, i \neq j$.
- (iii) For all $f \in L^2(\mathcal{F})$,

$$(2.4) \quad P_\ell^{(q)}(f) \in L^2(\mathcal{F}_\ell^{(q)}) \ominus L^2(\mathcal{F}_{\ell-1}^{(q)}), q = 1, \dots, d, \ell \in \mathbb{Z},$$

and

$$P_j(f) \in \bigcap_{q=1}^d \left(L^2(\mathcal{F}_{j_q}^{(q)}) \ominus L^2(\mathcal{F}_{j_q-1}^{(q)}) \right) =: L_j^2, j \in \mathbb{Z}^d.$$

(iv) For all $i, j \in \mathbb{Z}^d, i \neq j$, and $f \in L^2(\mathcal{F})$, $P_i P_j f = 0$, almost surely.

Proof. (i) It suffices to show that for all $f \in L^2(\mathcal{F})$, $\ell_1, \ell_2 \in \mathbb{Z}, q_1, q_2 \in \{1, \dots, d\}, q_1 \neq q_2$,

$$\mathbb{E} \left[\mathbb{E} \left(f \mid \mathcal{F}_{\ell_1}^{(q_1)} \right) \mid \mathcal{F}_{\ell_2}^{(q_2)} \right] = \mathbb{E} \left[\mathbb{E} \left(f \mid \mathcal{F}_{\ell_2}^{(q_2)} \right) \mid \mathcal{F}_{\ell_1}^{(q_1)} \right].$$

This follows from (2.1) and (2.3).

(ii) Since $i \neq j$, without loss of generality, assume $i_1 > j_1$. Then,

$$\mathbb{E}(P_i(f) P_j(g)) = \mathbb{E} \mathbb{E} \left(P_i(f) P_j(g) \mid \mathcal{F}_{j_1}^{(1)} \right) = \mathbb{E} \left[P_j(g) \mathbb{E} \left(P_i(f) \mid \mathcal{F}_{j_1}^{(1)} \right) \right],$$

where the last step follows from the fact that $P_j(g)$ is $\mathcal{F}_{j_1}^{(1)}$ -measurable. By (i), $P_i(f)$ can be written as $P_{i_1}(g)$ for some $g \in L^2(\mathcal{F})$. Thus, since $\mathbb{E}(P_{i_1}(g) \mid \mathcal{F}_{i_1-1}^{(1)}) = 0$ and $i_1 - 1 \geq j_1$, we have $\mathbb{E}(P_i(f) \mid \mathcal{F}_{j_1}^{(1)}) = 0$ and thus the desired orthogonality.

(iii) The fact (2.4) follows from the definition. The other statement follows again from the commuting property.

(iv) It follows from (iii). \square

Orthomartingales can be obtained via commuting transformations. Namely, let $\{T_{e_q}\}_{q=1, \dots, d}$ be d bijective, bi-measurable and measure-preserving transformations on $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying in addition that $T_{e_q} \circ$

$T_{e_{q'}} = T_{e_{q'}} \circ T_{e_q}$ for all $q, q' = 1, \dots, d$. Then, $\{T_{e_q}\}_{q=1, \dots, d}$ generate a \mathbb{Z}^d -group of transformations $\{T_i\}_{i \in \mathbb{Z}^d}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{M} \subset \mathcal{F}$ be a σ -algebra on Ω such that $\mathcal{M} \subset T_{e_q}^{-1}\mathcal{M}$ for $q = 1, \dots, d$. In this way, $\mathcal{F}_i := T_i^{-1}\mathcal{M}, i \in \mathbb{Z}^d$ yield a commuting filtration.

Given commuting transformations $\{T_i\}_{i \in \mathbb{Z}^d}$, we consider stationary random fields of the form

$$\{X_i\}_{i \in \mathbb{Z}^d} \equiv \{f \circ T_i\}_{i \in \mathbb{Z}^d}$$

for some function f in the space $L^2(\mathcal{F})$. In particular, for any $f \in L^2(\mathcal{F})$,

$$\{(P_0 f) \circ T_i\}_{i \in \mathbb{N}^d}$$

gives a collection of stationary orthomartingale differences with respect to $\{\mathcal{F}_i\}_{i \in \mathbb{Z}_+^d}$.

Example 2.4. When $d = 1$, $P_j = \mathbb{E}_j - \mathbb{E}_{j-1}$ has been well studied. When $d = 2$,

$$P_{j_1, j_2} f = \mathbb{E}_{j_1, j_2} f - \mathbb{E}_{j_1, j_2-1} f - \mathbb{E}_{j_1-1, j_2} f + \mathbb{E}_{j_1-1, j_2-1} f.$$

We also write $U_i f = f \circ T_i, i \in \mathbb{Z}^d$. One can readily verify that $U_j P_i = P_{i+j} U_j$ for all $i, j \in \mathbb{Z}^d$. This identity will be useful in the sequel.

We conclude this section with two canonical examples for stationary orthomartingales.

Example 2.5. Let $\{\epsilon_i\}_{i \in \mathbb{Z}^d}$ be a collection of independent and identically distributed random variables with distribution μ . We will consider stationary fields as functions of $\{\epsilon_i\}_{i \in \mathbb{Z}^d}$. For this purpose, assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ has the following form

$$(2.5) \quad (\Omega, \mathcal{F}, \mathbb{P}) \equiv \left(\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}^{\mathbb{Z}^d}, \mu^{\mathbb{Z}^d} \right),$$

and we identify $\epsilon_i(\omega) = \omega_i, i \in \mathbb{Z}^d$. Let $\{T_i\}_{i \in \mathbb{Z}^d}$ be the \mathbb{Z}^d -group of shift operations of $\mathbb{R}^{\mathbb{Z}^d}$, so that $[T_i(\omega)]_j = \epsilon_{j+i}, i, j \in \mathbb{Z}^d$. It is straightforward to check that random variables $\{\epsilon_i\}_{i \in \mathbb{Z}^d}$ induce a commuting filtration $\{\mathcal{F}_i\}_{i \in \mathbb{Z}^d}$ by

$$(2.6) \quad \mathcal{F}_i = \sigma \{ \epsilon_j : j \in \mathbb{Z}^d, j \leq i \}, i \in \mathbb{Z}^d.$$

Example 2.6. Let $\{\epsilon_i^{(q)}\}_{i \in \mathbb{Z}}, q = 1, \dots, d$ be d independent collections of i.i.d. random variables. Consider $\mathcal{G}_i^{(q)} = \sigma(\epsilon_j^{(q)} : j \leq i), i \in \mathbb{Z}, q = 1, \dots, d$, and set $\mathcal{F}_i = \bigvee_{q=1}^d \mathcal{G}_{i_q}^{(q)}, i \in \mathbb{Z}^d$. Clearly this yields a commuting filtration and there is a natural class of commuting transformations.

3. A UNIFORM INTEGRABILITY RESULT

In this section, we establish a uniform integrability result for stationary orthomartingales (Lemma 3.1). This entails that the tightness of normalized stationary orthomartingales only requires finite second moment (Proposition 3.2), thus improving the result of Wang and Woodroffe [23]. In the sequel, we will apply Cairoli's maximal inequality [17, p. 19, Theorem 2.3.1] repeatedly: for an orthomartingale $\{M_n\}_{n \in \mathbb{N}^d}$ with respect to a commuting filtration $\{\mathcal{F}_n\}_{n \in \mathbb{N}^d}$,

$$(3.1) \quad \mathbb{E} \left(\max_{i \in [n]} |M_i| \right)^p \leq \left(\frac{p}{p-1} \right)^{dp} \mathbb{E} |M_n|^p,$$

for $p > 1$. To simplify the notation we write $\mathbb{E}|Z|^p \equiv \mathbb{E}(|Z|^p)$, $\mathbb{E}Z^{2k} \equiv \mathbb{E}(Z^{2k})$, and for $a > 0$, $\mathbb{E}_a Y^2 = \mathbb{E}(Y^2 \mathbf{1}_{\{Y^2 > a\}})$.

Lemma 3.1. *Let $\{M_n\}_{n \in \mathbb{N}^d}$ be a stationary orthomartingale with respect to a commuting filtration $\{\mathcal{F}_n\}_{n \in \mathbb{Z}_+^d}$. Suppose $\mathbb{E}D_1^2 < \infty$. Then,*

$$(3.2) \quad \lim_{a \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}_a \left(\max_{i \in [n]} \frac{|M_i|}{|n|^{1/2}} \right)^2 = 0.$$

Proof. Recall that $\{D_n\}_{n \in \mathbb{N}^d}$ are stationary orthomartingale differences. For each $i \in \mathbb{N}^d$, define

$$D_i(c) = P_i(D_i \mathbf{1}_{\{|D_i| \leq c\}}) \quad \text{and} \quad R_i(c) = D_i - D_i(c).$$

Clearly, $\{D_n(c)\}_{n \in \mathbb{N}^d}$ and $\{R_n(c)\}_{n \in \mathbb{N}^d}$ are still stationary orthomartingale differences and we write the corresponding orthomartingales by $\{M_n(c)\}_{n \in \mathbb{N}^d}$ and $\{M'_n(c)\}_{n \in \mathbb{N}^d}$, respectively. Then,

$$(3.3) \quad \mathbb{E}_a \left(\max_{i \in [n]} \frac{|M_i|}{|n|^{1/2}} \right)^2 \leq 4\mathbb{E}_{a/4} \left(\max_{i \in [n]} \frac{|M_i(c)|}{|n|^{1/2}} \right)^2 + 4\mathbb{E}_{a/4} \left(\max_{i \in [n]} \frac{|M'_i(c)|}{|n|^{1/2}} \right)^2.$$

Now, the first term on the right-hand side above can be bounded by

$$\begin{aligned} & 4 \left[\mathbb{E} \left(\max_{i \in [n]} \frac{|M_i(c)|}{|n|^{1/2}} \right)^4 \right]^{1/2} \times \mathbb{P}^{1/2} \left[\left(\max_{i \in [n]} \frac{|M_i(c)|}{|n|^{1/2}} \right)^2 > a/4 \right] \\ & \leq \frac{4}{|n|} \left[\mathbb{E} \left(\max_{i \in [n]} |M_i(c)| \right)^4 \right]^{1/2} \times \left(\frac{4}{a} \right)^{1/2} \left[\mathbb{E} \left(\max_{i \in [n]} \frac{|M_i(c)|}{|n|^{1/2}} \right)^2 \right]^{1/2}, \end{aligned}$$

which, by applying Cairoli's inequality (3.1) twice, can be bounded by

$$(3.4) \quad \frac{4}{|n|} \left(\frac{4}{3}\right)^{2d} (\mathbb{E} M_n^4(c))^{1/2} \times \left(\frac{2^{2d+2}}{a}\right)^{1/2} \frac{(\mathbb{E} M_n^2(c))^{1/2}}{|n|^{1/2}},$$

where the second term is bounded by $c(2^{2d+2}/a)^{1/2}$. The first term of (3.4) can be bounded by Kc^2 for some constant K depending only on d via Burkholder's inequality. To see this, first we observe that

$$\left\{ \sum_{i_1=1}^{n_1} \cdots \sum_{i_{d-1}=1}^{n_{d-1}} D_{i_1, \dots, i_{d-1}, i_d} \right\}_{i_d \in \mathbb{N}}$$

is a sequence of stationary martingale differences with respect to $\{\mathcal{F}_n^{(d)}\}_{n \in \mathbb{N}}$. Thus, Burkholder's inequality tells, for $p \geq 2$,

$$\|M_n\|_p = \left\| \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} D_i \right\|_p \leq C_p n_d^{1/2} \left\| \sum_{i_1=1}^{n_1} \cdots \sum_{i_{d-1}=1}^{n_{d-1}} D_i \right\|_p.$$

Repeating this argument, one obtains that

$$\mathbb{E}|M_n|^p \leq C_p^{dp} |n|^{p/2} \mathbb{E}|D_1|^p.$$

So the first term on the right-hand side of (3.3) can be bounded by $Kc^3/a^{1/2}$ for some constant K depending only on d .

Next, the second term on the right-hand side of (3.3) can be bounded by

$$\begin{aligned} 4\mathbb{E} \left(\max_{i \in [n]} \frac{|M'_i(c)|}{|n|^{1/2}} \right)^2 \\ \leq 2^{2+2d} \mathbb{E} (P_1 D_1 \mathbf{1}_{\{|D_1| > c\}})^2 \leq 2^{2+2d} \mathbb{E} (D_1^2 \mathbf{1}_{\{|D_1| > c\}}). \end{aligned}$$

Combing all above, the desired result (3.2) follows. \square

An immediate consequence of Lemma 3.1 is the tightness of normalized stationary orthomartingales. For $t \in \mathbb{R}^d, n \in \mathbb{N}^d$, we write $t \cdot n = (t_1 n_1, \dots, t_d n_d)$ and $M_t = M_{[t]}$.

Proposition 3.2. *Under the assumption of Lemma 3.1,*

$$\left\{ \frac{M_{t \cdot n}}{|n|^{1/2}} \right\}_{t \in [0,1]^d}$$

is tight in $D[0, 1]^d$. That is, for all $\epsilon > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\sup_{\substack{s, t \in [0, 1]^d \\ \|s - t\|_\infty < \delta}} |M_{s \cdot n} - M_{t \cdot n}| > \epsilon \right) = 0.$$

Proof. For each $\delta \in (0, 1)$, $n \in \mathbb{N}^d$, write $\delta n = (\delta n_1, \dots, \delta n_d)$. Observe that

$$\begin{aligned} p_n(\epsilon, \delta) &:= \mathbb{P} \left(\sup_{\substack{s, t \in [0, 1]^d \\ \|s - t\|_\infty < \delta}} |M_{s \cdot n} - M_{t \cdot n}| > 3^d \epsilon \right) \\ &\leq \sum_{i \in \{0, \dots, \lfloor 1/\delta \rfloor\}^d} \mathbb{P} \left(\sup_{t \in [\delta]^d} \frac{|M_{(\delta i) \cdot n} - M_{(\delta i + t) \cdot n}|}{|n|^{1/2}} > \epsilon \right) \\ &\leq \frac{2^d}{\delta^d} \mathbb{P} \left(\max_{i \in [\delta n]} \frac{|M_i|}{|n|^{1/2}} > \epsilon \right). \end{aligned}$$

Now,

$$\begin{aligned} \mathbb{P} \left(\max_{i \in [\delta n]} \frac{|M_i|}{|n|^{1/2}} > \epsilon \right) &\leq \frac{1}{\epsilon^2} \mathbb{E}_{\epsilon^2} \left(\max_{i \in [\delta n]} \frac{|M_i|}{|n|^{1/2}} \right)^2 \\ &= \frac{\delta^d}{\epsilon^2} \mathbb{E}_{\epsilon^2/\delta^d} \left(\max_{i \in [\delta n]} \frac{|M_i|}{|\delta n|^{1/2}} \right)^2. \end{aligned}$$

So,

$$\limsup_{n \rightarrow \infty} p_n(\epsilon, \delta) \leq \frac{2}{\epsilon^2} \limsup_{n \rightarrow \infty} \mathbb{E}_{\epsilon^2/\delta^d} \left(\max_{i \in [n]} \frac{|M_i|}{|n|^{1/2}} \right)^2.$$

The proof is completed by applying (3.2). \square

4. ORTHOMARTINGALE COBOUNDARY REPRESENTATION

In this section, we extend the notion of martingale coboundary representation [13, 16, 22] to orthomartingales. For $S \subset \{1, \dots, d\}$, write $S^c = \{1, \dots, d\} \setminus S$.

Proposition 4.1. *For $f \in L^2(\mathcal{F})$ satisfying, for some $M \in \mathbb{N}$,*

$$(4.1) \quad \mathbb{E}_{-M}^{(q)} f = 0 \quad \text{and} \quad f - \mathbb{E}_M^{(q)} f = 0 \quad \text{for all} \quad q = 1, \dots, d,$$

one can write

$$(4.2) \quad f = \sum_{S \subset \{1, \dots, d\}} \prod_{q \in S^c} (I - U_{e_q}) h_S$$

for some functions $\{h_S\}_{S \subset \{1, \dots, d\}}$, with the convention $\prod_{q \in \emptyset} (I - U_{e_q}) \equiv I$, satisfying for each $S \subset \{1, \dots, d\}$,

$$(4.3) \quad h_S \in \bigcap_{q \in S} L^2(\mathcal{F}_0^{(q)}) \ominus L^2(\mathcal{F}_{-1}^{(q)}),$$

and

$$(4.4) \quad h_{\{1, \dots, d\}} = \sum_{j \in \mathbb{Z}^d} P_0 U_j f.$$

The property (4.3) tells that $\{U_{e_q}^k h_S\}_{k \in \mathbb{N}}$ forms a sequence of stationary martingale differences with respect to $\{\mathcal{F}_n^{(q)}\}_{n \in \mathbb{N}}$ for $q \in S$. The explicit formula of h_S is given below in (4.6).

Example 4.2. In the case $d = 1$, (4.2) reads as

$$f = h_{\{1\}} + (I - U)h_\emptyset,$$

which is the coboundary decomposition in dimension one. In the case $d = 2$, (4.2) reads as

$$(4.5) \quad f = h_{\{1,2\}} + (I - U_{1,0})h_{\{2\}} + (I - U_{0,1})h_{\{1\}} + (I - U_{0,1})(I - U_{1,0})h_\emptyset,$$

where m is an orthomartingale difference with respect to $\{\mathcal{F}_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$, and $h_{0,1}$ and $h_{1,0}$ are martingale differences with respect to $\{\mathcal{F}_{\infty,j}\}_{j \in \mathbb{Z}}$, $\{\mathcal{F}_{i,\infty}\}_{i \in \mathbb{Z}}$, respectively.

Proof of Proposition 4.1. We construct h_S , $S \subset \{1, \dots, d\}$ by induction. For $i \in \mathbb{Z}$, write $v(i) = \mathbf{1}_{\{i < 0\}}$. Write $\mathbb{Z}_1 = \{i \in \mathbb{Z} : i \geq 0\}$ and $\mathbb{Z}_0 = \{i \in \mathbb{Z} : i < 0\}$. Define two operators A_{e_q} and B_{e_q} by

$$A_{e_q} f = \sum_{i \in \mathbb{Z}} P_0^{(q)} U_{e_q}^i f$$

and

$$B_{e_q} f = \sum_{i \in \mathbb{Z}} (-1)^{v(i)+1} \sum_{k \in \mathbb{Z}_{v(i)}} P_i^{(q)} U_{e_q}^k f.$$

Clearly, for f satisfying the assumption (4.1), $A_{e_q} f$ and $B_{e_q} f$ are both well defined, and both as elements in $L^2(\mathcal{F})$ satisfy (4.1). It thus follows that compositions of operators A_{e_q} and B_{e_q} (e.g. (4.6) below) are well defined for functions satisfying (4.1). Observe also that all the pairs of operators $(A_{e_q}, A_{e_{q'}})$, $(A_{e_q}, B_{e_{q'}})$ and $(B_{e_q}, B_{e_{q'}})$ are commuting for $q \neq q'$ by definition.

Now, we show that in an orthomartingale coboundary representation of a function f under (4.1), one can choose h_S in (4.2) as

$$(4.6) \quad h_S = \prod_{r \in S} A_{e_r} \prod_{s \in S^c} B_{e_s} f.$$

The formula (4.2) with (4.6) is proved by induction. In the case $d = 1$, (4.2) becomes

$$f = A_{e_1}f + B_{e_1}f - U_{e_1}B_{e_1}f.$$

This is the decomposition developed in Volný [22], where $A_{e_1}f$ is a martingale difference and $B_{e_1}f - U_{e_1}B_{e_1}f$ is called the coboundary. Suppose one has shown for $d - 1$ and we now prove the case d . For $S \subset \{1, \dots, d-1\}$, write $S^c(d-1) = \{1, \dots, d-1\} \setminus S$. To apply the induction we view $\{\mathcal{F}_{i_1, \dots, i_{d-1}, \infty}\}_{i \in \mathbb{Z}^{d-1}}$ as a $(d-1)$ -dimensional commuting filtration. Thus one has

$$f = \sum_{S \subset \{1, \dots, d-1\}} g_S \text{ with } g_S = \prod_{q \in S^c(d-1)} (I - U_{e_q}) \prod_{r \in S} A_{e_r} \prod_{s \in S^c(d-1)} B_{e_s} f.$$

We apply the one-dimensional martingale coboundary decomposition to g_S , with respect to the filtration $\{\mathcal{F}_i^{(d)}\}_{i \in \mathbb{Z}}$. Indeed, one can verify $g_S = A_{e_d}g_S + (I - U_{e_d})B_{e_d}g_S$ with

$$\begin{aligned} A_{e_d}g_S &= A_{e_d} \prod_{q \in S^c(d-1)} (I - U_{e_q}) \prod_{r \in S} A_{e_r} \prod_{s \in S^c(d-1)} B_{e_s} f \\ &= \prod_{q \in S^c(d-1)} (I - U_{e_q}) A_{e_d} \prod_{r \in S} A_{e_r} \prod_{s \in S^c(d-1)} B_{e_s} f \\ &= \prod_{q \in (S \cup \{d\})^c} (I - U_{e_q}) \prod_{r \in S \cup \{d\}} A_{e_r} \prod_{s \in (S \cup \{d\})^c} B_{e_s} f \\ &= \prod_{q \in (S \cup \{d\})^c} (I - U_{e_q}) h_{S \cup \{d\}}, \end{aligned}$$

and

$$(I - U_{e_d})B_{e_d}g_S = \prod_{q \in S^c} (I - U_{e_q}) \prod_{r \in S} A_{e_r} \prod_{s \in S^c} B_{e_s} f = \prod_{q \in S^c} (I - U_{e_q}) h_S.$$

Thus,

$$\begin{aligned} f &= \sum_{S \subset \{1, \dots, d-1\}} \left[\prod_{q \in (S \cup \{d\})^c} (I - U_{e_q}) h_{S \cup \{d\}} + \prod_{q \in S^c} (I - U_{e_q}) h_S \right] \\ &= \sum_{S \subset \{1, \dots, d\}} \prod_{q \in S^c} (I - U_{e_q}) h_S. \end{aligned}$$

It remains to prove (4.3) and (4.4). Both follow from the construction (4.6), and the commuting property of involved operators. \square

Remark 4.3. Assumption (4.1) is enough for our purpose in the next section. Here we do not pursue a necessary and sufficient condition for the orthomartingale coboundary decomposition, as did in one dimension by Volný [22]. This would require more involved calculations and will be addressed elsewhere. A closely related recent result has been obtained by Gordin [14], who investigated the coboundary representation for *reversed* orthomartingales.

5. AN INVARIANCE PRINCIPLE

In this section, we prove the main result of the paper. Consider the probability space $(\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}_{\mathbb{R}}^{\mathbb{Z}^d}, \mu^{\mathbb{Z}^d})$ and the corresponding commuting transformations $\{T_i\}_{i \in \mathbb{Z}^d}$ and filtrations $\{\mathcal{F}_i\}_{i \in \mathbb{Z}^d}$ as described in Example 2.5. Consider a stationary random field $\{X_i\}_{i \in \mathbb{Z}^d}$ in form of

$$X_i = f \circ T_i(\{\epsilon_k\}_{k \in \mathbb{Z}^d}), i \in \mathbb{Z}^d.$$

We consider the following generalized Hannan's condition [15] for random fields

$$(5.1) \quad \sum_{i \in \mathbb{Z}^d} \|P_0 X_i\|_2 < \infty.$$

Consider partial sums $S_n = \sum_{i \in [n]}, n \in \mathbb{N}$. Under (5.1), there exists $D_0 \in L^2$ such that

$$\sum_{i \in \mathbb{Z}^d} P_0 X_i \text{ converges to } D_0 \text{ in } L^2.$$

That is, for all $m, n \in \mathbb{N}^d$, $\|\sum_{i \in [-m, n]} P_0 X_i - D_0\|_2 \rightarrow 0$ as $m, n \rightarrow \infty$.

Theorem 5.1. *Consider a stationary random field $\{X_i\}_{i \in \mathbb{Z}^d}$ described as above with zero mean. If Hannan's condition (5.1) holds, then,*

$$\left\{ \frac{S_{[n \cdot t]}(f)}{|n|^{1/2}} \right\}_{t \in [0, 1]^d} \Rightarrow \{\sigma \mathbb{B}_t\}_{t \in [0, 1]^d}$$

as $n \rightarrow \infty$ in $D([0, 1]^d)$, where $\{\mathbb{B}_t\}_{t \in [0, 1]^d}$ is a standard Brownian sheet and $\sigma^2 = \mathbb{E}D_0^2$.

Proof of Theorem 5.1. The idea of proof is by orthomartingale approximation. Now introduce, for each $n \in \mathbb{N}^d$,

$$(5.2) \quad M_n = \sum_{i \in [n]} D_i \quad \text{with} \quad D_i = U_i D_0, i \in \mathbb{N}^d.$$

One easily sees that $\{M_n\}_{n \in \mathbb{N}^d}$ is a d -parameter orthomartingale with respect to the filtration $\{\mathcal{F}_i\}_{i \in \mathbb{Z}^d}$.

It has been established that for an orthomartingale $\{M_n\}_{n \in \mathbb{N}^d}$ with stationary orthomartingale differences with respect to the filtration generated by i.i.d. random variables,

$$(5.3) \quad \left\{ \frac{M_{\lfloor n \cdot t \rfloor}(f)}{|n|^{1/2}} \right\}_{t \in [0,1]^d} \Rightarrow \{\sigma \mathbb{B}_t\}_{t \in [0,1]^d}$$

in $D([0,1]^d)$. For the convergence of finite-dimensional distributions, see Wang and Woodroffe [23]; for the tightness under finite second moment, see Proposition 3.2. Thus, it suffices to show,

$$(5.4) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{1}{|n|^{1/2}} \max_{m \in [n]} |S_m(f) - M_m| > \epsilon \right) = 0 \text{ for all } \epsilon > 0.$$

To do so, observe that $f \in L^2(\mathcal{F})$ and the fact that $\{\epsilon_i\}_{i \in \mathbb{Z}^d}$ are i.i.d. imply

$$(5.5) \quad f = \sum_{i \in \mathbb{Z}^d} P_i f$$

where the summation converges in L^2 , and introduce

$$f^{(k)} = \sum_{i \in [-k,k]} P_i f$$

and

$$M_n^{(k)} = \sum_{i \in [n]} U_i D_0^{(k)} \quad \text{with} \quad D_0^{(k)} = \sum_{i \in [-k,k]} P_0 U_i f.$$

Then,

$$(5.6) \quad \begin{aligned} & \max_{m \in [n]} |S_m(f) - M_m| \\ & \leq \max_{m \in [n]} |S_m(f) - S_m(f^{(k)})| + \max_{m \in [n]} |S_m(f^{(k)}) - M_m^{(k)}| + \max_{m \in [n]} |M_m^{(k)} - M_m|. \end{aligned}$$

We control the three maxima separately.

(i) To estimate the first term on the right-hand side of (5.6), we need the following maximal inequality.

Lemma 5.2. *Under the assumption of Theorem 5.1, for all $n \in \mathbb{N}^d$,*

$$(5.7) \quad \left\| \max_{m \in [n]} S_m(f) \right\|_2 \leq 2^d |n|^{1/2} \sum_{i \in \mathbb{Z}^d} \|f_i\|_2,$$

with $f_i = P_0 U_i f \in L_0^2, i \in \mathbb{Z}^d$.

The proof is postponed to the end of section. Now, (5.7) yields

$$(5.8) \quad \mathbb{P} \left(\frac{1}{|n|^{1/2}} \max_{m \in [n]} |S_m(f) - S_m(f^{(k)})| > \epsilon \right) \leq \left(\frac{2^d}{\epsilon} \sum_{i \in \mathbb{Z}^d} \|(f - f^{(k)})_i\|_2 \right)^2.$$

Since $U_i P_j = P_{j+i} U_i$, $i, j \in \mathbb{Z}^d$, observe that

$$(f - f^{(k)})_i = P_0 U_i \left(\sum_{j \notin [-k, k]} P_j f \right) = P_0 \sum_{j \notin [-k, k]} P_{j+i} U_i f = f_i \mathbf{1}_{\{i \notin [-k, k]\}}.$$

Thus, by taking $\min\{k_1, \dots, k_d\}$ large enough, the upper bound in (5.8) can be arbitrarily small.

(ii) To estimate the last term in the right-hand side of (5.6), observe that $\{M_n - M_n^{(k)}\}_{n \in \mathbb{N}^d}$ is still a stationary orthomartingale. Again by Cairoli's maximal inequality, we have

$$(5.9) \quad \mathbb{P} \left(\frac{1}{|n|^{1/2}} \max_{m \in [n]} |M_m - M_m^{(k)}| > \epsilon \right) \leq \left(\frac{2^d}{\epsilon} \|D_0 - D_0^{(k)}\|_2 \right)^2.$$

Thus, by taking $\min\{k_1, \dots, k_d\}$ large enough, the upper bound in (5.9) can be arbitrarily small.

(iii) At last, write

$$S_m(f^{(k)}) - M_m^{(k)} = \sum_{i \in [m]} U_i(f^{(k)} - D_0^{(k)}).$$

It remains to show that

$$(5.10) \quad \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\frac{1}{|n|^{1/2}} \max_{m \in [n]} \left| \sum_{i \in [m]} U_i(f^{(k)} - D_0^{(k)}) \right| > \epsilon \right) = 0.$$

By Proposition 4.1, $f^{(k)} - D_0^{(k)}$ has an orthomartingale coboundary representation (4.2), and in particular, (4.4) becomes

$$\begin{aligned} h_{\{1, \dots, d\}} &= P_0 \sum_{j \in \mathbb{Z}^d} U_j(f^{(k)} - D_0^{(k)}) \\ &= P_0 \sum_{j \in \mathbb{Z}^d} U_j \sum_{\ell \in [-k, k]} P_\ell f - P_0 \sum_{j \in \mathbb{Z}^d} U_j \sum_{\ell \in [-k, k]} P_0 U_\ell f \\ &= P_0 \sum_{\ell \in [-k, k]} U_\ell f - P_0 \sum_{\ell \in [-k, k]} U_\ell f = 0. \end{aligned}$$

Thus,

$$(5.11) \quad f^{(k)} - D_0^{(k)} = \sum_{S \subseteq \{1, \dots, d\}} \prod_{q \in S^c} (I - U_{e_q}) h_S.$$

To prove (5.10), it suffices to show for each $S \subsetneq \{1, \dots, d\}$,

$$(5.12) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left(\max_{m \in [n]} \frac{\sum_{i \in [m]} U_i (\prod_{q \in S^c} (I - U_{e_q}) h_S)}{|n|^{1/2}} > \epsilon \right) = 0.$$

To better illustrate, we first prove the case $d = 2$. Suppose $S = \{1\}$. Notice that $U_{1,0}^k = U_{k,0}$, $k \in \mathbb{Z}$ by definition, and similarly for $U_{0,1}$. Then, for $n \in \mathbb{N}^2$,

$$(5.13) \quad \begin{aligned} & \max_{m \in [n]} \sum_{i \in [m]} U_i \prod_{q \in S^c} (I - U_{e_q}) h_S \\ &= \max_{m \in [n]} \sum_{i \in [m]} U_{1,0}^{i_1} (U_{0,1}^{i_2} (I - U_{0,1}) h_1) = \max_{m \in [n]} \sum_{i_1=1}^{m_1} U_{1,0}^{i_1} (U_{0,1} - U_{0,m_2+1}) h_1 \\ &\leq 2 \max_{m_2=1, \dots, n_2+1} \left| U_{0,m_2} \max_{m_1=1, \dots, n_1} \sum_{i_1=1}^{m_1} U_{i_1,0} h_1 \right|. \end{aligned}$$

Write

$$\widetilde{M}_{m_1} = \sum_{i_1=1}^{m_1} U_{i_1,0}^{i_1} h_1.$$

Observe that by Proposition 4.1, $\{U_{1,0}^{i_1} h_1\}_{i_1 \in \mathbb{N}}$ is a sequence of stationary martingale differences with respect to the filtration $\{\mathcal{F}_n^{(1)}\}_{n \in \mathbb{N}}$. So, the probability in (5.12) is bounded by

$$(5.14) \quad \begin{aligned} & (n_2 + 1) \mathbb{P} \left(\frac{|\max_{m_1 \leq n_1} \widetilde{M}_{m_1}|}{(n_1 n_2)^{1/2}} > \epsilon/2 \right) \\ &\leq (n_2 + 1) \mathbb{E}_{\frac{\epsilon^2}{4}} \left(\frac{\max_{m_1 \leq n_1} \widetilde{M}_{m_1}^2}{n_1 n_2} \right) \leq \frac{n_2 + 1}{n_2 \epsilon^2} \mathbb{E}_{n_2 \frac{\epsilon^2}{4}} \left(\frac{\max_{m_1 \leq n_1} \widetilde{M}_{m_1}}{n_1^{1/2}} \right)^2. \end{aligned}$$

By uniform integrability (3.2), the last term above tends to zero as $\min(n_1, n_2) \rightarrow \infty$.

The same argument applies to the case $S = \{2\}$. For the case $S = \emptyset$, the probability in (5.12) is bounded by

$$(5.15) \quad \mathbb{P} \left(\frac{\max_{m_1 \leq n_1, m_2 \leq n_2} U_{m_1, m_2} h_S}{(n_1 n_2)^{1/2}} > \epsilon/4 \right) \\ \leq (n_1 + 1)(n_2 + 1) \mathbb{P} \left(\frac{|h_S|}{(n_1 n_2)^{1/2}} > \epsilon/4 \right) \leq \frac{(n_1 + 1)(n_2 + 1)}{n_1 n_2 \epsilon^2 / 16} \mathbb{E}_{n_1 n_2 \epsilon^2 / 16} h_S^2,$$

which tends to zero as $n \rightarrow \infty$. We have thus proved (5.12) for $d = 2$.

At last we sketch the proof for general $d \geq 3$. Without loss of generality, we suppose $S^c = \{s+1, \dots, d\}$ with $s = 0, \dots, d-1$. In the case $s = 0$, (5.15) can be easily generalized and we omit the details. In the case $s \geq 1$, observe that

$$\begin{aligned} & \sum_{i \in [m]} U_i \prod_{q \in S^c} (I - U_{e_q}) h_S \\ &= \sum_{i_1=1}^{m_1} \cdots \sum_{i_s=1}^{m_s} \prod_{q=1}^s U_{e_q}^{i_q} \sum_{i_{s+1}=1}^{m_{s+1}} \cdots \sum_{i_d=1}^{m_d} \prod_{q=s+1}^d U_{e_q}^{i_q} \prod_{r=s+1}^d (I - U_{e_r}) h_S \\ &= \sum_{i_1=1}^{m_1} \cdots \sum_{i_s=1}^{m_s} \prod_{q=1}^s U_{e_q}^{i_q} \prod_{r=s+1}^d (U_{e_r} - U_{e_r}^{m_r+1}) h_S \\ &= \prod_{r=s+1}^d (U_{e_r} - U_{e_r}^{m_r+1}) \left(\sum_{i_1=1}^{m_1} \cdots \sum_{i_s=1}^{m_s} \prod_{q=1}^s U_{e_q}^{i_q} h_S \right). \end{aligned}$$

Thus, (5.14) becomes, for $n \in \mathbb{N}^d$,

$$(5.16) \quad \prod_{q=s+1}^d (n_q + 1) \mathbb{P} \left(\frac{\max_{m_q \leq n_q, q=1, \dots, s} \widetilde{M}_{m_1, \dots, m_s}}{|n|^{1/2}} > \epsilon/2^{d-s} \right),$$

with

$$\widetilde{M}_{m_1, \dots, m_s} = \sum_{i_1=1}^{m_1} \cdots \sum_{i_s=1}^{m_s} \prod_{q=1}^s U_{e_q}^{i_q} h_S.$$

By Proposition 4.1 again, this time $\{\prod_{q=1}^s U_{e_q}^{i_q} h_S\}_{i_1, \dots, i_s \in \mathbb{N}^s}$ form a collection of s -dimension stationary orthomartingale differences, with respect to the commuting filtration $\{\mathcal{F}_{n_1, \dots, n_s, \infty, \dots, \infty}\}_{n_1, \dots, n_s \in \mathbb{N}^s}$. Therefore (5.16) can be bounded as before by

$$\prod_{q=s+1}^d \frac{n_q + 1}{n_q} \mathbb{E}_{(\prod_{q=s+1}^d n_q) \epsilon^2 / 2^{2(d-s)}} \left(\frac{\max_{m_q \leq n_q, q=1, \dots, s} \widetilde{M}_{m_1, \dots, m_s}}{(\prod_{q=1}^s n_q)^{1/2}} \right)^2,$$

which again tends to zero as $n \rightarrow \infty$ by the uniform integrability (3.2). \square

Remark 5.3. The approximation of S_n by M_n (5.4) actually holds for more general commuting filtrations, possibly with extra assumption on the regularity of the random field ($\mathbb{E}(X_0 \mid \mathcal{F}_i) = 0$ whenever $\min_{q=1,\dots,d} i_q = -\infty$ and X_0 is $\mathcal{F}_{\infty,\dots,\infty}$ -measurable) so that (5.5) holds.

However, a crucial ingredient of the proof is the invariance principle (5.3) for M_n established by Wang and Woodroffe [23]. For this result to hold, our assumption on the underlying random field of i.i.d. random variables indexed by \mathbb{Z}^d is needed. Without this assumption, in general a stationary orthomartingale difference random field may converge to a limit distribution that is not Gaussian [23, Example 1].

Proof of Lemma 5.2. Recall that $f_i = P_0 U_i f$ and (5.5). Since

$$S_n(f) = \sum_{j \in [n]} U_j \sum_{i \in \mathbb{Z}^d} P_i f = \sum_{i \in \mathbb{Z}^d} \sum_{j \in [n]} U_j P_i f = \sum_{i \in \mathbb{Z}^d} \sum_{j \in [n]} U_{j-i} f_i,$$

we have for all $m \in [n]$,

$$\begin{aligned} |S_m(f)| &\leq \sum_{i \in \mathbb{Z}^d} \left| \sum_{j \in [m]} U_{j-i} f_i \right| \\ &\leq \sum_{i \in \mathbb{Z}^d} \max_{k \in [n]} \left| \sum_{j \in [k]} U_{j-i} f_i \right| \leq \sum_{i \in \mathbb{Z}^d} U_{-i} \left(\max_{k \in [n]} \left| \sum_{j \in [k]} U_j f_i \right| \right). \end{aligned}$$

Therefore,

$$\left\| \max_{m \in [n]} S_m(f) \right\|_2 \leq \sum_{i \in \mathbb{Z}^d} \left\| \max_{k \in [n]} \left| \sum_{j \in [k]} U_j f_i \right| \right\|_2.$$

Observe that for each i fixed, $\{\sum_{j \in [k]} U_j f_i\}_{k \in \mathbb{N}^d}$ is an orthomartingale with respect to the filtration $\{\mathcal{F}_i\}_{i \in \mathbb{Z}^d}$. Therefore, by Cairoli's inequality (3.1),

$$\left\| \max_{k \in [n]} \left| \sum_{j \in [k]} U_j f_i \right| \right\|_2 \leq 2^d \|S_n(f_i)\|_2 = 2^d |n|^{1/2} \|f_i\|_2,$$

where in the last step we used the fact that $\{U_j f_i\}_{j \in [n]}$ is a collection of stationary orthomartingale differences. \square

6. DISCUSSIONS

There are some recent developments on sufficient conditions for central limit theorem and invariance principle of stationary random fields, notably by El Machkouri et al. [11] and Wang and Woodroffe [23]. We compare our condition to theirs.

We first show that the Hannan's condition is *strictly* weaker than Wu's condition [24, 11]

$$(6.1) \quad \sum_{i \in \mathbb{Z}^d} \delta_i(f) < \infty$$

where $\delta_i(f)$ is the *physical dependence measure* for a stationary random field $\{f \circ T_i\}_{i \in \mathbb{Z}^d}$, which we will recall in a moment. El Machkouri et al. [11] showed that this condition implies central limit theorem for stationary random fields. In dimension one, it has been shown in Wu [24, Theorem 1] that (6.1) implies Hannan's condition (5.1), and the argument can be easily adapted to high dimension and the details are omitted. We provide an example in Proposition 6.1 below that satisfies Hannan's condition but not (6.1). It suffices to construct a martingale difference random field that violates (6.1).

However, we remark also that the results of El Machkouri et al. [11] are more general in the sense that they include central limit theorem and invariance principle for random fields indexed by non-rectangular sets. In this case they assume stronger assumption on the moment in terms of entropy of the index sets.

In the sequel, suppose $\epsilon = \{\epsilon_i\}_{i \in \mathbb{Z}^d}$ is a sequence of i.i.d. random variables with $\mathbb{P}(\epsilon_0 = \pm 1) = 1/2$. Then, for a function $f : \{\pm 1\}^{\mathbb{Z}^d} \rightarrow \mathbb{R}$, the physical dependence measure is defined by

$$(6.2) \quad \delta_i = \|f(\epsilon) - f(\epsilon^{*i})\|_2 \quad \text{with} \quad \epsilon_k^{*i} = \begin{cases} \epsilon_k & \text{if } k \neq i \\ \epsilon_i^* & \text{if } k = i, \end{cases}$$

where ϵ^{*i} is a copy of ϵ_k , independent of ϵ .

Proposition 6.1. *Under the above assumption, there exists a martingale difference that does not satisfy (6.1).*

Proof. We first address the case $d = 1$. Set

$$\begin{aligned} Z_1(\epsilon) &= \mathbf{1}_{\{\epsilon_{-2}=-1\}} \mathbf{1}_{\{\epsilon_{-1}=-1\}} \epsilon_0 \\ Z_2(\epsilon) &= \mathbf{1}_{\{\epsilon_{-4}=-1\}} \mathbf{1}_{\{\epsilon_{-3}=-1\}} \mathbf{1}_{\{\epsilon_{-2}=1\}} \mathbf{1}_{\{\epsilon_{-1}=1\}} \epsilon_0 \\ &\dots \\ Z_n(\epsilon) &= \mathbf{1}_{\{\epsilon_{-2n}=-1\}} \mathbf{1}_{\{\epsilon_{-2n+1}=-1\}} \mathbf{1}_{\{\epsilon_{-2n+2}=1\}} \cdots \mathbf{1}_{\{\epsilon_{-1}=1\}} \epsilon_0, n \geq 3. \end{aligned}$$

Define

$$f = f(\epsilon) = \sum_{n=1}^{\infty} c_n Z_n(\epsilon)$$

for certain sequence of real values $\{c_n\}_{n \in \mathbb{N}}$ such that

$$(6.3) \quad \sum_{n=1}^{\infty} c_n^2 \|Z_n(\epsilon)\|_2^2 < \infty.$$

Under this condition, clearly f is well defined and a martingale difference in the sense that $f \in \mathcal{F}_0$ and $\mathbb{E}(f \mid \mathcal{F}_{-1}) = 0$.

Now we compute δ_i defined in (6.2). Observe that for $i > 0$, $\delta_i = 0$. From now on suppose $i < 0$. Suppose $i = -(2k-1)$ or $-2k$ for some $k \in \mathbb{N}$, then we have

$$f(\epsilon) - f(\epsilon^{*i}) = \sum_{j=k}^{\infty} c_j (Z_j(\epsilon) - Z_j(\epsilon^{*i})).$$

Observe that by construction, for all $j \neq j'$, $Z_j(\epsilon) Z_{j'}(\epsilon^{*i}) \equiv 0$, and

$$\mathbb{P}(Z_j(\epsilon) \neq Z_j(\epsilon^{*i}) \mid Z_j(\epsilon) \neq 0) = 1/2, \text{ for all } j \geq k.$$

Thus,

$$\left[\sum_{j=k}^{\infty} c_j (Z_j(\epsilon) - Z_j(\epsilon^{*i})) \right]^2 = \sum_{j=k}^{\infty} c_j^2 (Z_j(\epsilon) - Z_j(\epsilon^{*i}))^2,$$

and for each $j \geq k$,

$$\begin{aligned} \mathbb{E}(Z_j(\epsilon) - Z_j(\epsilon^{*i}))^2 &\geq \mathbb{P}(Z_j(\epsilon) \neq 0) \mathbb{E}[(Z_j(\epsilon) - Z_j(\epsilon^{*i}))^2 \mid Z_j(\epsilon) \neq 0] \\ &= \mathbb{P}(Z_j(\epsilon) \neq 0) \mathbb{P}(Z_j(\epsilon) \neq Z_j(\epsilon^{*i}) \mid Z_j(\epsilon) \neq 0) \\ &= \frac{1}{2} \|Z_j(\epsilon)\|_2^2. \end{aligned}$$

Thus,

$$\delta_i^2 = \mathbb{E} \left[\sum_{j=k}^{\infty} c_j (Z_j(\epsilon) - Z_j(\epsilon^{*i}))^2 \right]^2 \geq \frac{1}{2} \sum_{j=k}^{\infty} c_j^2 \|Z_j(\epsilon)\|_2^2,$$

and

$$\sum_{i \leq -1} \delta_i^2(f) \geq \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} c_j^2 \|Z_j(\epsilon)\|_2^2 = \sum_{j=1}^{\infty} j c_j^2 \|Z_j(\epsilon)\|_2^2.$$

Now, choose $\{c_n\}_{n \in \mathbb{N}}$ such that $c_n^2 \|Z_n(\epsilon)\|_2^2 = n^{-2}$, so that f is well-defined since (6.3) is satisfied. However, $\sum_i \delta_i^2(f) = \infty$ whence $\sum_i \delta_i(f) = \infty$, as desired.

It remains to prove the case $d \geq 2$. This can be done by first assigning an ordering of the space $\{i \in \mathbb{Z}^d : i \leq -\mathbf{1}\}$ and then embedding the one-dimensional construction. The details are omitted. \square

Next, our results also improve Wang and Woodroffe [23]. They proved a central limit theorem for stationary random field under the condition

$$(6.4) \quad \sum_{k \in \mathbb{N}^d} \frac{\|\mathbb{E}(X_k | \mathcal{F}_0)\|_2}{|k|^{1/2}} < \infty,$$

and established an invariance principle under a slightly stronger assumption, replacing $\|\cdot\|_2$ by $\|\cdot\|_p$ for some $p > 2$ in (6.4). The Hannan's condition (5.1) we assumed here is weaker than (6.4). This is known in dimension one, see Peligrad and Utev [20, Corollary 2]. We prove the result for high dimension in Lemma 6.2.

Lemma 6.2. *Condition (6.4) implies Hannan's condition (5.1).*

Proof. For $n \in \mathbb{N}^d$, set $a_n = \|P_0 X_n\|_2$. Then, it is equivalent to show

$$(6.5) \quad \sum_{n \in \mathbb{N}^d} a_n = \infty \text{ implies } \sum_{n \in \mathbb{N}^d} \frac{1}{|n|^{1/2}} \left(\sum_{k \geq n} a_k^2 \right)^{1/2} = \infty.$$

To see this, first observe that by orthogonality, for each $n \in \mathbb{N}^d$,

$$\|\mathbb{E}(X_n | \mathcal{F}_0)\|_2^2 = \sum_{k \geq \mathbf{0}} \|P_{-k} X_n\|_2^2 = \sum_{k \geq n} \|P_0 X_k\|_2^2 = \sum_{k \geq n} a_k^2.$$

To prove (6.5), introduce $B_n = \{k \in \mathbb{N}^d : n \leq k \leq 2n - \mathbf{1}\}$, and observe

$$\begin{aligned} \sum_{n \in \mathbb{N}^d} \frac{1}{|n|^{1/2}} \left(\sum_{k \geq n} a_k^2 \right)^{1/2} &\geq \sum_{n \in \mathbb{N}^d} \frac{1}{|n|^{1/2}} \left(\sum_{k \in B_n} a_k^2 \right)^{1/2} \\ &\geq \sum_{n \in \mathbb{N}^d} \frac{1}{|n|} \sum_{k \in B_n} a_k = \sum_{k \in \mathbb{N}^d} a_k \sum_{n \in \mathbb{N}^d} \frac{1}{|n|} \mathbf{1}_{\{k \in B_n\}} \geq \frac{1}{2^d} \sum_{k \in \mathbb{N}^d} a_k. \end{aligned}$$

\square

The fact that (6.4) is actually *strictly stronger* than Hannan's condition follows from Durieu and Volný [10] and Durieu [9], in the case $d = 1$. Indeed, they constructed a counterexample to show that Hannan's condition does not imply the Maxwell–Woodroffe condition [18], and the latter is known to be strictly weaker than (6.4). Thus, if Hannan's condition implies (6.4), it then implies the Maxwell–Woodroffe condition, hence a contradiction. The counterexample therein can be generalized to \mathbb{Z}^d .

Acknowledgement The authors thank Mohamed El Machkouri and Davide Giraudo for many inspiring discussions. The second author thanks Laboratoire Mathématiques Raphaël Salem at Université de Rouen for the invitation during June, 2013, during which the main result of this work was obtained.

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